## Definition (Variation)

Let $f : [a,b] \to \mathbb{R}$ be a function and $P = \{a = x_0 < \cdots < x_n = b\}$ be a partition of $[a,b]$. The variation of $f$ with respect to $P$ is defined as $V(f,P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$ and its total variation is $V(f) = \sup_P V(f,P)$.

A function $f$ has bounded variation if $V(f) < \infty$.

$C[a,b] = \{\text{continuous functions on } [a,b]\}$

$B[a,b] = \{\text{bounded functions on } [a,b]\}$

$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$.

$BV[a,b] = \{\text{functions of bounded variation on } [a,b]\}$

$\|f\| = V(f)$

Claim: $C[a,b]^* = BV[a,b]$

Definition (Riemann-Stieltjes Integral)

Let $f \in C[a,b]$, $g \in BV[a,b]$ and $P = \{a = x_0 < \cdots < x_n < b\}$ be a partition with tags $t_k \in [x_{k-1}, x_k]$. The Riemann-Stieltjes sum of $f$ with respect to $g$ and $P$ is defined as

$S(f, g, P) = \sum_{k=1}^{n} f(t_k)(g(x_k) - g(x_{k-1}))$. The Riemann-Stieltjes Integral of $f$ with respect to $g$ is defined as

$$\int_a^b f(x)\, dg(x) = \lim_{\|P\| \to 0} S(f, g, P).$$

Rmk: The existence of the limit is similar to the standard Riemann integral using the fact that $f$ is uniformly continuous and $g$ has bounded variation.

Proof of $C[a,b]^* = BV[a,b]$:

Define $T: BV[a,b] \to C[a,b]^*$ by

$$(Tg)(f) = \int_a^b f(x)\, dg(x)$$

for any $f \in C[a,b]$ and $g \in BV[a,b]$.

$T$ is well-defined by the existence of Riemann-Stieltjes Integral and clearly linear.

(i) $T$ is injective.

$$Tg = 0 \Rightarrow \int_a^b f(x)\,dg(x) = 0 \quad \text{for all } f \in C[a,b]$$

$$\Rightarrow \int_a^b s(x)\,dg(x) = 0 \quad \text{for all step function } s$$

For any $a \le c < d \le b$, let $s(x) = \begin{cases} 1, & c \le x \le d, \\ 0, & \text{otherwise.} \end{cases}$

Choosing a partition $P$ with $x_{k_1} = c$ and $x_{k_2} = d$ for some $k_1, k_2$, we get $g(d) - g(c) = \sum_{k=k_1+1}^{k_2} g(x_k) - g(x_{k-1}) < \varepsilon$ for any $\varepsilon > 0$. Thus $g(d) - g(c) = 0$ for any $a \le c < d \le b$.

Hence $g \equiv 0$.

(ii) $T$ is surjective.

Pick any $L \in C[a,b]^*$. We want to find $g \in BV[a,b]$ such that $L = Tg$.

Notice that $C[a,b]$ is a subspace of $B[a,b]$

By Hahn–Banach Theorem, $L$ has an extension $\tilde{L}$ on $B[a,b]$ with $\|L\|_{C[a,b]^*} = \|\tilde{L}\|_{B[a,b]^*}$

Define $g$ by $g(x) = \tilde{L}\chi_{[a,x]}$.

$\vdash_1$: $g \in BV[a,b]$.

Pf: Pick any partition $P = \{a = x_0 < \cdots < x_n = b\}$

$$\sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| = \sum_{k=1}^{n} \varepsilon_k (g(x_k) - g(x_{k-1})) \qquad \varepsilon_k = \pm 1$$

$$= \sum_{k=1}^{n} \varepsilon_k (\tilde{L}\chi_{[a,x_k]} - \tilde{L}\chi_{[a,x_{k-1}]})$$

$$= \tilde{L}\left(\sum_{k=1}^{n} \varepsilon_k (\chi_{[a,x_k]} - \chi_{[a,x_{k-1}]})\right)$$

$$\leq \|\tilde{L}\|_{B[a,b]^*} \left\| \sum_{k=1}^{n} \varepsilon_k (\chi_{[a,x_k]} - \chi_{[a,x_{k-1}]}) \right\|_\infty$$

$$= \|\tilde{L}\|_{B[a,b]^*}$$

$$= \|L\|_{C[a,b]^*}.$$

$\vdash_2$: $L(f) = T_g(f)$ for all $f \in C[a,b]$.

Pf: Pick any $f \in C[a,b]$. Fix $\varepsilon > 0$. By uniform continuity of $f$, $\exists\ \delta_1 > 0$ such that

$|f(x) - f(y)| < \varepsilon$ for $x, y \in [a,b]$ with $|x-y| < \delta_1$.

By definition of Riemann-Stieltjes integral,

$\exists\ \delta_2 > 0$ such that for all partition $P$ with

$\|P\| < \delta_2$ and $t_k = x_k$, $\left| T_g(f) - S(f, g, P) \right| < \varepsilon$.

Take a partition $P$ with $\|P\| < \min\{\delta_1, \delta_2\}$.

Put $\hat{f} = \sum_{k=1}^{n} f(x_k) \chi_{[x_{k-1}, x_k]}$.

Then $\|f - \hat{f}\|_\infty < \varepsilon$ and $\tilde{L}(\hat{f}) = S(f, g, P)$.

Therefore, $\left| T_g(f) - L(f) \right|$

$$\leq \left| T_g(f) - \tilde{L}(\hat{f}) \right| + \left| \tilde{L}(\hat{f}) - L(f) \right|$$

$$= \left| \int_a^b f(x)\, dg(x) - S(f, g, P) \right| + \left| \tilde{L}(\hat{f} - f) \right|$$

$$< \varepsilon + \|\tilde{L}\|_{B[a,b]^*} \|\hat{f} - f\| \varepsilon$$

Letting $\varepsilon \to 0$ gives $T_g(f) = L(f)$.

(iii) $T$ is an isometry.

Pick any $f \in C[a,b]$, $g \in BV[a,b]$ and partition $P$ with tags $t_k$.

$$|S(f,g,P)| = \left| \sum_{k=1}^{n} f(t_k)\left(g(x_k) - g(x_{k-1})\right) \right|$$

$$\leq \sum_{k=1}^{n} |f(t_k)| \, |g(x_k) - g(x_{k-1})|$$

$$\leq \|f\|_\infty \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})|$$

$$\leq \|f\|_\infty V(g)$$

Thus $|T_g(f)| \leq \|f\|_\infty V(f)$ for any $f \in C[a,b]$, i.e.,

$\|Tg\|_{C[a,b]^*} \leq V(g)$.

By proof of (ii), $\exists\ \tilde{g} \in BV[a,b]$ such that

$V(\tilde{g}) \leq \|Tg\|_{C[a,b]^*}$ and $Tg = T\tilde{g}$.

By (i), $g = \tilde{g}$.

Hence $V(g) \leq \|Tg\|_{C[a,b]^*}$

$\square$